PARTIALLY ORDERED SETS IN MAPLE

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If we define a binary operation on a set, then a partial order relation on the set is one where the relation is reflexive, antisymmetric and transitive. Maple provides a number of tools for dealing with finite partially ordered sets, or *posets* for short. These are all contained in the PartiallyOrderedSets package, which must be included in the worksheet.

To define a partial ordered set, we need a set and a binary relation. A set in Maple is defined by listing its elements within curly braces. A binary relation can be defined using a function. An example is as follows (centred lines are Maple's responses):

```
with(PartiallyOrderedSets):
divis := (x, y) -> irem(y, x) = 0;
S := {1, 2, 3, 4, 5, 6};
                        S := {1, 2, 3, 4, 5, 6}
poS2 := PartiallyOrderedSet(S, divis);
                  poS2 := < a poset with 6 elements >
```

The `irem(y,x)` function computes $y \mod x$, that is, the remainder when y is divided by x. Thus `poS2` is the partially ordered set constructed from `S` with the binary relation $x*y$ such that y is divisible by x.

We can visualize a partially ordered set by drawing a graph, using `DrawGraph(poS2)`. See Fig. 1.

The graph doesn't show all the ordered pairs, since the 1 vertex is connected to all other vertices (since 1 divides everything). The graph shows all ordered pairs in their simplest form, that is, if $a*b$ and $b*c$ then (from transitivity) we also have $a*c$, but the graph will show only $a*b$ and $b*c$ (assuming there are no other elements between a and b, and between b and c). In this case, a is said to be *adjacent* to b (and b is adjacent to c). We can get a list of all adjacent pairs with the command:

```
AdjacencyList(poS2);
                  [{1,2,3,5} {2,4,6} {3,6} {4} {5} {6}]
```

In each grouping, the first number is the left-hand element in the relation, and the other numbers are those to which it is related. Longer chains can

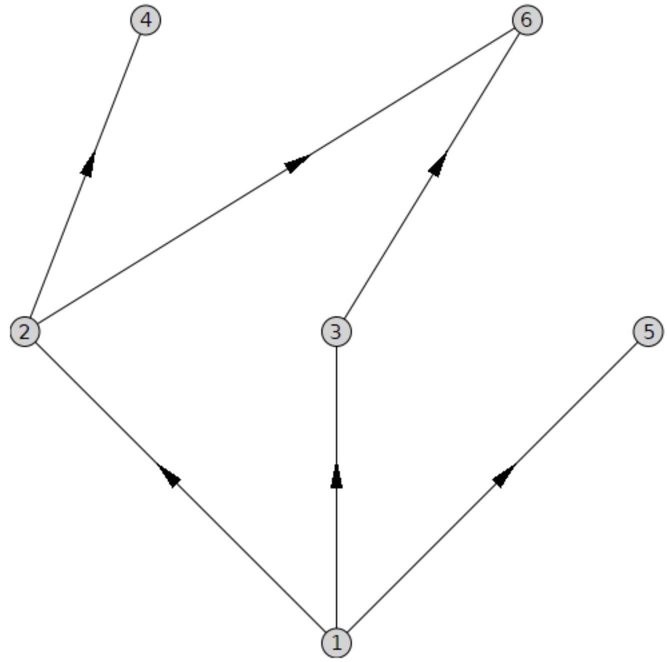

FIGURE 1. Graph of partially ordered set poS2.

be worked out using transitivity, which essentially means there is a path through the graph from the first element to its counterpart.

As another example, consider

```
Y := {1, 2, 3, 4, 5, 10, 12, 15, 24, 30, 36, 45, 60};
        Y := {1, 2, 3, 4, 5, 10, 12, 15, 24, 30, 36, 45, 60}
poY1 := PartiallyOrderedSet(Y, divis);
                  poY1 := < a poset with 13 elements >
```

This uses the same binary operation as above (divisibility) and generates the graph in Fig. 2.

The *maximal elements* are those at the ends of paths in the graph, and can be found using:

```
MaximalElements(poY1);
                              {24, 36, 45, 60}
```

Maple will check that the conditions for a partial order are satisfied when you attempt to create the set. For example, if we tried using the binary relation

```
divis1 := (x, y) -> irem(y, x) = 1;
```

That is, where $y \mod x = 1$, we get an error when we try to construct

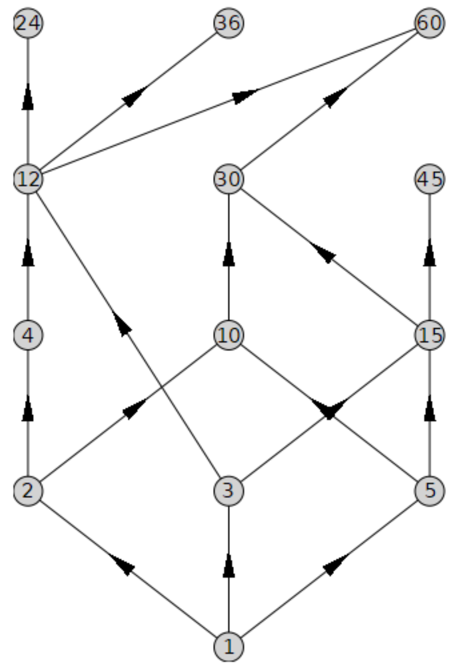

FIGURE 2. Graph of `poY1`.

```
poY2 := PartiallyOrderedSet(Y, divis1);
```

The error message states “non-transitivity detected”. We can see that this relation is not transitive, since $5 \mod 2 = 1$ and $36 \mod 5 = 1$ but $36 \mod 2 = 0$. However, we can try constructing a partially ordered set using the set `Y2` and relation `divis1`:

```
Y2 := {2, 5, 11};
```

$$Y2 := \{2, 5, 11\}$$

We are successful since now $5 \mod 2 = 1$, $11 \mod 5 = 1$ and $11 \mod 2 = 1$, so the relation is transitive on this set.

Maple provides many other functions dealing with partially ordered sets, but we’ll leave these until later.